KUWAIT UNIVERSITY

Department of Mathematics

Math 102 July 24, 2010

Calculus II Midterm 2 Time: 90 minutes

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1. (a) Use integration by parts to show that

$$\int \frac{(1-\ln x)}{x^2}\,dx = \frac{\ln x}{x} + C\,. \qquad \text{[2 marks]}$$

(b) Determine the convergence or divergence of the improper integrals

(i) $\displaystyle\int_0^1 \frac{(1-\ln x)}{x^2}\,dx\,,$ (ii) $\displaystyle\int_1^\infty \frac{(1-\ln x)}{x^2}\,dx\,.$ [1½ marks each]

2. Evaluate the following integrals. [3 marks each]

(a) $\displaystyle\int \frac{1}{\sqrt{1-e^{-x}}}\,dx$

(b) $\displaystyle\int \sqrt{6x-5-x^2}\,dx$

(c) $\displaystyle\int \sin^3 x \sec^{1/3} x\,dx$

(d) $\displaystyle\int \frac{3x^2+1}{(x+1)^2(x^2+1)}\,dx$

(e) $\displaystyle\int \frac{\csc x}{2\csc x+1}\,dx$

3. (a) Verify that

$$\int \sec^3\theta\,d\theta = \tfrac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right] + C\,. \qquad \text{[2 marks]}$$

(b) Use the substitution $x = \tan^4\theta$ to evaluate the integral

$$\int \frac{1}{\sqrt{x+\sqrt{x}}}\,dx\,. \qquad \text{[3 marks]}$$

[25 marks total]

SOLUTIONS

1. (a) Let $u = 1 - \ln x$ and $dv = (1/x^2)\,dx$. Then $du = (-1/x)\,dx$ and $v = -1/x$. So that

$$\begin{aligned}\int \frac{1-\ln x}{x^2}\,dx &= (1-\ln x)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\left(-\frac{1}{x}\right)dx \\ &= \frac{\ln x - 1}{x} - \int \frac{1}{x^2}\,dx = \frac{\ln x - 1}{x} + \frac{1}{x} + C = \frac{\ln x}{x} + C.\end{aligned}$$

(b) (i)

$$\int_t^1 \frac{1-\ln x}{x^2}\,dx = \left.\frac{\ln x}{x}\right|_t^1 = -\frac{\ln t}{t} \to \infty \quad \text{as } t \to 0^+.$$

Therefore, $\displaystyle\int_0^1 \frac{1-\ln x}{x^2}\,dx$ is divergent.

(ii)

$$\int_1^t \frac{1-\ln x}{x^2}\,dx = \left.\frac{\ln x}{x}\right|_1^t = \frac{\ln t}{t} \to 0 \quad \text{as } t \to \infty.$$

Therefore, $\displaystyle\int_1^\infty \frac{1-\ln x}{x^2}\,dx$ is convergent.

2. (a) Substitute[1] $t = \sqrt{1-\mathrm{e}^{-x}}$. Then $x = -\ln(1-t^2)$ and $dx = [2t/(1-t^2)]\,dt$. So that,

$$\begin{aligned}\int \frac{1}{\sqrt{1-\mathrm{e}^{-x}}}\,dx &= \int \frac{1}{t}\,\frac{2t}{1-t^2}\,dt = \int \frac{2}{1-t^2}\,dt = \int \left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt \\ &= \ln|1+t| - \ln|1-t| + C \\ &= \ln\left(1+\sqrt{1-\mathrm{e}^{-x}}\right) - \ln\left(1-\sqrt{1-\mathrm{e}^{-x}}\right) + C.\end{aligned}$$

(b) Completing the square,

$$6x - 5 - x^2 = -(x^2 - 6x + 5) = -[(x-3)^2 - 4] = 4 - (x-3)^2.$$

Substitute $x - 3 = 2\sin\theta$ which implies that $\sqrt{6x-5-x^2} = 2\cos\theta$ and $dx = 2\cos\theta\,d\theta$. This gives

$$\begin{aligned}\int \sqrt{6x-5-x^2}\,dx &= 4\int \cos^2\theta\,d\theta = 2\int (1+\cos 2\theta)\,d\theta = 2\theta + \sin 2\theta + C \\ &= 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\arcsin\left(\frac{x-3}{2}\right) + \frac{x-3}{2}\sqrt{6x-5-x^2} + C.\end{aligned}$$

(c)

$$\begin{aligned}\int \sin^3 x \sec^{1/3} x\,dx &= \int (\sin x)(1-\cos^2 x)\cos^{-1/3} x\,dx \\ &= \int (\cos^{5/3} x - \cos^{-1/3} x)(-\sin x)\,dx \\ &= \tfrac{3}{8}\cos^{8/3} x - \tfrac{3}{2}\cos^{2/3} x + C.\end{aligned}$$

[1] An alternative substitution is $\mathrm{e}^{-x/2} = \sin\theta$ which implies that $\sqrt{1-\mathrm{e}^{-x}} = \cos\theta$ and $dx = -2\cot\theta\,d\theta$. So that $\int \frac{1}{\sqrt{1-\mathrm{e}^{-x}}}\,dx = -2\int \csc\theta\,d\theta = 2\ln|\csc\theta + \cot\theta| + C = \cdots$.

(d) The partial fraction decomposition is

$$\frac{3x^2+1}{(x+1)^2\,(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

where

$$\begin{aligned} 3x^2+1 &= A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 \\ &= (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + A+B+D. \end{aligned}$$

Equating coefficients,

$$\left\{ \begin{aligned} A+C &= 0 \\ A+B+2C+D &= 3 \\ A+C+2D &= 0 \\ A+B+D &= 1 \end{aligned} \right. \qquad \Longrightarrow \qquad \left\{ \begin{aligned} A &= -1 \\ B &= 2 \\ C &= 1 \\ D &= 0. \end{aligned} \right.$$

Hence,

$$\begin{aligned} \int \frac{3x^2+1}{(x+1)^2\,(x^2+1)}\,dx &= \int \left[\frac{-1}{x+1} + \frac{2}{(x+1)^2} + \frac{x}{x^2+1} \right] dx \\ &= -\ln|x+1| - \frac{2}{x+1} + \frac{1}{2}\ln(x^2+1) + C. \end{aligned}$$

(e) Use Weierstrass substitution, i.e. $t = \tan(x/2)$ which implies that $\csc x = (1+t^2)/2t$ and $dx = [2/(1+t^2)]\,dt$. Then,

$$\begin{aligned} \int \frac{\csc x}{2\csc x + 1}\,dx &= \int \frac{[(1+t^2)/2t]}{2[(1+t^2)/2t]+1}\,\frac{2}{1+t^2}\,dt = \int \frac{1}{t^2+t+1}\,dt \\ &= \int \frac{1}{(t+1/2)^2 + 3/4}\,dt = \frac{2}{\sqrt{3}}\arctan\left(\frac{t+1/2}{\sqrt{3}/2}\right) + C \\ &= \frac{2}{\sqrt{3}}\arctan\left(\frac{2\tan(x/2)+1}{\sqrt{3}}\right) + C. \end{aligned}$$

3. (a) By the Fundamental Theorem of Calculus, it suffices to show that the derivative of the right-hand side is the integrand, i.e.

$$\begin{aligned} &\frac{d}{d\theta}\left(\tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\Big]\right) \\ &= \tfrac{1}{2}\Big[(\sec\theta\tan\theta)\tan\theta + (\sec\theta)\sec^2\theta + \sec\theta\Big] = \tfrac{1}{2}(\sec\theta)(\tan^2\theta + \sec^2\theta + 1) \\ &= \tfrac{1}{2}(\sec\theta)(\sec^2\theta - 1 + \sec^2\theta + 1) = \sec^3\theta. \end{aligned}$$

(b) The substitution $x = \tan^4\theta$ for $0 < \theta < \pi/2$ gives $\sqrt{x+\sqrt{x}} = \sqrt{\tan^4\theta + \tan^2\theta} = \sec\theta\tan\theta$ and $dx = 4\tan^3\theta\sec^2\theta\,d\theta$. Hence,

$$\begin{aligned} \int \frac{1}{\sqrt{x+\sqrt{x}}}\,dx &= \int \frac{1}{\sec\theta\tan\theta}\,4\tan^3\theta\sec^2\theta\,d\theta = 4\int \tan^2\theta\sec\theta\,d\theta \\ &= 4\int (\sec^3\theta - \sec\theta)\,d\theta \\ &= 2\,(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|) - 4\ln|\sec\theta+\tan\theta| + C \\ &= 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| + C \\ &= 2\sqrt{x+\sqrt{x}} - 2\ln\left(\frac{\sqrt{x+\sqrt{x}}}{\sqrt[4]{x}} + \sqrt[4]{x}\right) + C. \end{aligned}$$